

# Modular forms, modular symbols

(PARI-GP version 2.13.4)

## Modular Forms

### Dirichlet characters

Characters are encoded in three different ways:

- a `t_INT`  $D \equiv 0, 1 \bmod 4$ : the quadratic character  $(D/\cdot)$ ;
- a `t_INTMOD`  $\text{Mod}(m, q)$ ,  $m \in (\mathbf{Z}/q)^*$  using a canonical bijection with the dual group (the Conrey character  $\chi_q(m, \cdot)$ );
- a pair  $[G, \text{chi}]$ , where  $G = \text{znstar}(q, 1)$  encodes  $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$  and the vector  $\text{chi} = [c_1, \dots, c_k]$  encodes the character such that  $\chi(g_j) = e(c_j/d_j)$ .

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	<code>G = znstar(<math>q, 1</math>)</code>
convert datum $D$ to $[G, \chi]$	<code>znchar(<math>D</math>)</code>
Galois orbits of Dirichlet characters	<code>chargalois(<math>G</math>)</code>

### Spaces of modular forms

Arguments of the form  $[N, k, \chi]$  give the level weight and nebentypus  $\chi$ ;  $\chi$  can be omitted:  $[N, k]$  means trivial  $\chi$ .

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mfinit(<math>[N, k, \chi], 0</math>)</code>
initialize $S_k(\Gamma_0(N), \chi)$	<code>mfinit(<math>[N, k, \chi], 1</math>)</code>
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$	<code>mfinit(<math>[N, k, \chi], 2</math>)</code>
initialize $E_k(\Gamma_0(N), \chi)$	<code>mfinit(<math>[N, k, \chi], 3</math>)</code>
initialize $M_k(\Gamma_0(N), \chi)$	<code>mfinit(<math>[N, k, \chi]</math>)</code>
find eigenforms	<code>mfsplit(<math>M</math>)</code>
statistics on self-growing caches	<code>getcache()</code>

We let $M = \text{mfinit}(\dots)$ denote a modular space.	
describe the space $M$	<code>mfdescribe(<math>M</math>)</code>
recover $(N, k, \chi)$	<code>mfparams(<math>M</math>)</code>
... the space identifier (0 to 4)	<code>mfspace(<math>M</math>)</code>
... the dimension of $M$ over $\mathbf{C}$	<code>mfdim(<math>M</math>)</code>
... a $\mathbf{C}$ -basis $(f_i)$ of $M$	<code>mfbasis(<math>M</math>)</code>
... a basis $(F_j)$ of eigenforms	<code>mfeigenbasis(<math>M</math>)</code>
... polynomials defining $\mathbf{Q}(\chi)/(F_j)/\mathbf{Q}(\chi)$	<code>mffields(<math>M</math>)</code>

matrix of Hecke operator $T_n$ on $(f_i)$	<code>mfheckemat(<math>M, n</math>)</code>
eigenvalues of $w_Q$	<code>mfatkineigenvalues(<math>M, Q</math>)</code>
basis of period poynomials for weight $k$	<code>mfperiodpolbasis(<math>k</math>)</code>
basis of the Kohnen $+$ -space	<code>mfkohnenbasis(<math>M</math>)</code>
... new space and eigenforms	<code>mfkohneneigenbasis(<math>M, b</math>)</code>
isomorphism $S_k^+(4N, \chi) \rightarrow S_{2k-1}(N, \chi^2)$	<code>mfkohnenbijection(<math>M</math>)</code>

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mfdim(<math>[N, k, \chi]</math>)</code>
dimension of $S_k(\Gamma_0(N), \chi)$	<code>mfdim(<math>[N, k, \chi], 1</math>)</code>
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$	<code>mfdim(<math>[N, k, \chi], 2</math>)</code>
dimension of $M_k(\Gamma_0(N), \chi)$	<code>mfdim(<math>[N, k, \chi], 3</math>)</code>
dimension of $E_k(\Gamma_0(N), \chi)$	<code>mfdim(<math>[N, k, \chi], 4</math>)</code>
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	<code>mfsturm(<math>N, k</math>)</code>
$\Gamma_0(N)$ <b>cosets</b>	
list of right $\Gamma_0(N)$ cosets	<code>mfcosets(<math>N</math>)</code>
identify coset a matrix belongs to	<code>mfcoset</code>

### Cusps

a cusp is given by a rational number or oo.

lists of cusps of $\Gamma_0(N)$	<code>mfcusps(<math>N</math>)</code>
number of cusps of $\Gamma_0(N)$	<code>mfnumcusps(<math>N</math>)</code>
width of cusp $c$ of $\Gamma_0(N)$	<code>mfcuspswidth(<math>N, c</math>)</code>
is cusp $c$ regular for $M_k(\Gamma_0(N), \chi)$ ?	<code>mfcuspsisregular(<math>[N, k, \chi], c</math>)</code>

### Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients	<code>mftobasis(<math>\text{mf}, \text{vec}</math>)</code>
There are also many predefined ones:	
Eisenstein series $E_k$ on $Sl_2(\mathbf{Z})$	<code>mfEk(<math>k</math>)</code>
Eisenstein-Hurwitz series on $\Gamma_0(4)$	<code>mfEH(<math>k</math>)</code>
unary $\theta$ function (for character $\psi$ )	<code>mfTheta(<math>\{\psi\}</math>)</code>
Ramanujan's $\Delta$	<code>mfDelta()</code>
$E_k(\chi)$	<code>mfeisenstein(<math>k, \chi</math>)</code>
$E_k(\chi_1, \chi_2)$	<code>mfeisenstein(<math>k, \chi_1, \chi_2</math>)</code>
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	<code>mffrometaquo(<math>a</math>)</code>
newform attached to ell. curve $E/\mathbf{Q}$	<code>mffromell(<math>E</math>)</code>
identify an $L$ -function as a eigenform	<code>mffromlfun(<math>L</math>)</code>
$\theta$ function attached to $Q > 0$	<code>mffromqt(<math>Q</math>)</code>
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mftraceform(<math>[N, k, \chi]</math>)</code>
trace form in $S_k(\Gamma_0(N), \chi)$	<code>mftraceform(<math>[N, k, \chi], 1</math>)</code>

### Operations on modular forms

In this section,  $f, g$  and the  $F[i]$  are modular forms

$f \times g$	<code>mfmul(<math>f, g</math>)</code>
$f/g$	<code>mfddiv(<math>f, g</math>)</code>
$f^n$	<code>mfpow(<math>f, n</math>)</code>
$f(q)/q^v$	<code>mfshift(<math>f, v</math>)</code>
$\sum_{i \leq k} \lambda_i F[i]$ , $L = [\lambda_1, \dots, \lambda_k]$	<code>mflinear(<math>F, L</math>)</code>
$f = g?$	<code>mfisequal(<math>f, g</math>)</code>
expanding operator $B_d(f)$	<code>mfbd(<math>f, d</math>)</code>
Hecke operator $T_n f$	<code>mfhecke(<math>mf, f, n</math>)</code>
initialize Atkin-Lehner operator $w_Q$	<code>mfatkininit(<math>mf, Q</math>)</code>
... apply $w_Q$ to $f$	<code>mfatkin(<math>w_Q, f</math>)</code>
twist by the quadratic char $(D/\cdot)$	<code>mftwist(<math>f, D</math>)</code>
derivative wrt. $q \cdot d/dq$	<code>mfderiv(<math>f</math>)</code>
see $f$ over an absolute field	<code>mfreltoabs(<math>f</math>)</code>
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	<code>mfderivE2(<math>f</math>)</code>
Rankin-Cohen bracket $[f, g]_n$	<code>mfbracket(<math>f, g, n</math>)</code>
Shimura lift of $f$ for discriminant $D$	<code>mfshimura(<math>mf, f, D</math>)</code>

### Properties of modular forms

In this section,  $f = \sum_n f_n q^n$  is a modular form in some space  $M$  with parameters  $N, k, \chi$ .

describe the form $f$	<code>mfdescribe(<math>f</math>)</code>
$(N, k, \chi)$ for form $f$	<code>mfparams(<math>f</math>)</code>
the space identifier (0 to 4) for $f$	<code>mfspace(<math>mf, f</math>)</code>
$[f_0, \dots, f_n]$	<code>mfcoefs(<math>f, n</math>)</code>
$f_n$	<code>mfcoef(<math>f, n</math>)</code>
is $f$ a CM form?	<code>mfisCM(<math>f</math>)</code>
is $f$ an eta quotient?	<code>mfisetaquo(<math>f</math>)</code>
Galois rep. attached to all $(1, \chi)$ eigenforms	<code>mfgaloisistype(<math>M</math>)</code>
... single eigenform	<code>mfgaloisistype(<math>M, F</math>)</code>
... as a polynomial fixed by $\text{Ker } \rho_F$	<code>mfgaloisprojrep(<math>M, F</math>)</code>
decompose $f$ on <code>mfbasis</code> ( $M$ )	<code>mftobasis(<math>M, f</math>)</code>
smallest level on which $f$ is defined	<code>mfconductor(<math>M, f</math>)</code>
decompose $f$ on $\oplus S_k^{\text{new}}(\Gamma_0(d))$ , $d \mid N$	<code>mftonew(<math>M, f</math>)</code>
valuation of $f$ at cusp $c$	<code>mfcuspsval(<math>M, f, c</math>)</code>
expansion at $\infty$ of $f \mid_k \gamma$	<code>mfslashexpansion(<math>M, f, \gamma, n</math>)</code>
$n$ -Taylor expansion of $f$ at $i$	<code>mftaylor(<math>f, n</math>)</code>
all rational eigenforms matching criteria	<code>mfeigensearch</code>
... forms matching criteria	<code>mfsearch</code>

### Forms embedded into C

Given a modular form  $f$  in  $M_k(\Gamma_0(N), \chi)$  its field of definition  $Q(f)$  has  $n = [Q(f) : Q(\chi)]$  embeddings into the complex numbers. If  $n = 1$ , the following functions return a single answer, attached to the canonical embedding of  $f$  in  $\mathbf{C}[[q]]$ ; else a vector of  $n$  results, corresponding to the  $n$  conjugates of  $f$ .

complex embeddings of $Q(f)$	<code>mfembed(<math>f</math>)</code>
... embed coefs of $f$	<code>mfembed(<math>f, v</math>)</code>
evaluate $f$ at $\tau \in \mathcal{H}$	<code>mfeval(<math>f, \tau</math>)</code>
$L$ -function attached to $f$	<code>lfunmf(<math>mf, f</math>)</code>
... eigenforms of new space $M$	<code>lfunmf(<math>M</math>)</code>

### Periods and symbols

The functions in this section depend on  $[Q(f) : Q(\chi)]$  as above.

initialize symbol $fs$ attached to $f$	<code>mfsymbol(<math>M, f</math>)</code>
evaluate symbol $fs$ on path $p$	<code>mfsymboleval(<math>fs, p</math>)</code>
Petersson product of $f$ and $g$	<code>mfpetersson(<math>fs, gs</math>)</code>
period polynomial of form $f$	<code>mfperiodpol(<math>M, fs</math>)</code>
period polynomials for eigensymbol $FS$	<code>mfmanin(<math>FS</math>)</code>

Modular Symbols

Let  $G = \Gamma_0(N)$ ,  $V_k = \mathbf{Q}[X, Y]_{k-2}$ ,  $L_k = \mathbf{Z}[X, Y]_{k-2}$ . We let  $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$ ; an element of  $\Delta$  is a *path* between cusps of  $X_0(N)$  via the identification  $[b] - [a] \rightarrow$  the path from  $a$  to  $b$ . A path is coded by the pair  $[a, b]$ , where  $a, b$  are rationals or  $\infty$ , denoting the point at infinity ( $1 : 0$ ).

Let  $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$ ; an element of  $\mathbf{M}_k(G)$  is a  $V_k$ -valued *modular symbol*. There is a natural decomposition  $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$  under the action of the  $*$  involution, induced by complex conjugation. The `msinit` function computes either  $\mathbf{M}_k$  ( $\varepsilon = 0$ ) or its  $\pm$ -parts ( $\varepsilon = \pm 1$ ) and fixes a minimal set of  $\mathbf{Z}[G]$ -generators ( $g_i$ ) of  $\Delta$ .

initialize  $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$   
the level  $M$   
the weight  $k$   
the sign  $\varepsilon$   
Farey symbol attached to  $G$   
... attached to  $H < G$   
 $H \backslash G$  and right  $G$ -action  
 $\mathbf{Z}[G]$ -generators ( $g_i$ ) and relations for  $\Delta$   
decompose  $p = [a, b]$  on the ( $g_i$ )

`msinit( $N, k, \{\varepsilon = 0\}$ )`  
`msgetlevel( $M$ )`  
`msgetweight( $M$ )`  
`msgetsign( $M$ )`  
`mspolygon( $M$ )`  
`msfarey( $F, inH$ )`  
`mscosets( $genG, inH$ )`  
`mspathgens( $M$ )`  
`mspathlog( $M, p$ )`

Create a symbol

Eisenstein symbol attached to cusp  $c$   
cuspidal symbol attached to  $E/\mathbf{Q}$   
symbol having given Hecke eigenvalues  
is  $s$  a symbol ?

`msfromcusp( $M, c$ )`  
`msfromell( $E$ )`  
`msfromhecke( $M, v, \{H\}$ )`  
`msissymbol( $M, s$ )`

Operations on symbols

the list of all  $s(g_i)$   
evaluate symbol  $s$  on path  $p = [a, b]$   
Petersson product of  $s$  and  $t$

`mseval( $M, s$ )`  
`mseval( $M, s, p$ )`  
`mspetersson( $M, s, t$ )`

Operators on subspaces

An operator is given by a matrix of a fixed  $\mathbf{Q}$ -basis.  $H$ , if given, is a stable  $\mathbf{Q}$ -subspace of  $\mathbf{M}_k(G)$ : operator is restricted to  $H$ .  
matrix of Hecke operator  $T_p$  or  $U_p$   
matrix of Atkin-Lehner  $w_Q$   
matrix of the  $*$  involution

`mshecke( $M, p, \{H\}$ )`  
`msatkinlehner( $M, Q\{H\}$ )`  
`msstar( $M, \{H\}$ )`

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a  $\mathbf{Q}$ -basis. If  $H$  is a Hecke-stable subspace of  $M_k(G)^+$  or  $M_k(G)^-$ , it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform  $\sum_n a_n q^n$ .  
cuspidal subspace  $S_k(G)^\varepsilon$   
Eisenstein subspace  $E_k(G)^\varepsilon$   
new part of  $S_k(G)^\varepsilon$   
split  $H$  into simple subspaces (of  $\dim \leq d$ )  
dimension of a subspace  
( $a_1, \dots, a_B$ ) for attached newform  
 $\mathbf{Z}$ -structure from  $H^1(G, L_k)$  on subspace  $A$

`mscuspidal( $M$ )`  
`mseisenstein( $M$ )`  
`msnew( $M$ )`  
`mssplit( $M, H, \{d\}$ )`  
`msdim( $M$ )`  
`msqexpansion( $M, H, \{B\}$ )`  
`mslattice( $M, A$ )`

Overconvergent symbols and  $p$ -adic  $L$  functions

Let  $M$  be a full modular symbol space given by `msinit` and  $p$  be a prime. To a classical modular symbol  $\phi$  of level  $N$  ( $v_p(N) \leq 1$ ), which is an eigenvector for  $T_p$  with nonzero eigenvalue  $a_p$ , we can attach a  $p$ -adic  $L$ -function  $L_p$ . The function  $L_p$  is defined on continuous characters of  $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ ; in GP we allow characters  $\langle \chi \rangle^{s_1} \tau^{s_2}$ , where  $(s_1, s_2)$  are integers,  $\tau$  is the Teichmüller character and  $\chi$  is the cyclotomic character.

The symbol  $\phi$  can be lifted to an *overconvergent* symbol  $\Phi$ , taking values in spaces of  $p$ -adic distributions (represented in GP by a list of moments modulo  $p^n$ ).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that  $v_p(a_p) = 0$  if *flag* = 0 (fastest), and that  $v_p(a_p) \geq \textit{flag}$  otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to  $\Phi$  allowing to compute  $L_p$  to high accuracy.

initialize  $Mp$  to lift symbols  
lift symbol  $\phi$   
eval overconvergent symbol  $\Phi$  on path  $p$   
*mu* for  $p$ -adic  $L$ -functions  
 $L_p^{(r)}(\chi^s)$ ,  $s = [s_1, s_2]$   
 $\hat{L}_p(\tau^i)(x)$

`mspadicinit( $M, p, n, \{\textit{flag}\}$ )`  
`mstooms( $Mp, \phi$ )`  
`msomseval( $Mp, \Phi, p$ )`  
`mspadicmoments( $Mp, S, \{D = 1\}$ )`  
`mspadicL( $mu, \{s = 0\}, \{r = 0\}$ )`  
`mspadicseries( $mu, \{i = 0\}$ )`

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